ON FINSLERIAN HYPERSURFACE WITH GENERALIZED (α, β)-METRIC

Pradeep Kumar[1] and Madhu T. S.[2]

[1]Department of Mathematics, School of Engineering, Presidency University, Itgalpura, Bengaluru-560064, Karnataka, INDIA.

[2] Department of Mathematics, Sri Jagadguru Renukacharya College of Science, Arts and Commerce, Anand Rao Circle, Bengaluru-560009, Karnataka, INDIA.

mail Id: pradeepget@gmail.com and madhuts2327@gmail.com

Abstract: The purpose of the present paper is to investigate cetrain geometrical properties of hypersurface of a Finsler space with generalized (α, β)-metric $L = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$, where μ_1, μ_2 *and* μ_3 are constants and $\mu_1 \neq 0$. Further, we prove this hypersurface to be a hyperplane of 1^{st} and 2^{nd} kind.

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1. Introduction

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space, i.e., an n-dimensional differential manifold M^n equipped with a fundamental function $L(x, y)$. The concept of an (α, β)-metric $L(\alpha, \beta)$ was introduced by M. Matsumoto [7] and has been studied by many authors [3, 4, 18]. A Finsler metric $L(x, y)$ is called an (α, β)-metric $L(\alpha, \beta)$ if L is a positively homogeneous function of α and β of degree one, where $\alpha^2 = a_{ij}(x)y^iy^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M^n. A hypersurface M^{n-1} of the M^n represented by the equation $x^i = x^i(u^\alpha)$, $\alpha = 1, \cdots, n-1$, where u^α are Gaussian coordinates on M^{n-1}.

In 1985, the systematic theory of hypersurfaces is presented by M. Matsumoto [6]. He treated various types of Finslerian hypersurfaces and they are called hyperplane of 1^{st}, 2^{nd} and 3^{rd} kind. In 2001, I.Y-Lee, H.S. Park and Y.D. Lee [19] have obtained some results of the Finslerian hypersurface with metric $\alpha + \frac{\beta^2}{\alpha}$. M.K. Gupta and P.N. Pandey [2, 1] have contributed to the hypersurfaces of some special Finsler spaces. In 2009, H.G. Nagaraja, S.K. Narasimhamurthy, Pradeep Kumar and S.T. Aveesh obtained some geometrical properties of Finslerian hypersurfaces with (α, β)-metrics [16, 9]. Also, they have contributed a lot to the Finsler hypersurfaces [10, 14, 11, 8, 15].

In the present paper, we have studied cetrain geometrical properties of hypersurface of a Finsler space with generalized (α, β)-metric $L = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$, where $\mu_1, \ \mu_2 \ and \ \mu_3$ are constants and $\mu_1 \ \neq 0$. Also, we have proved this hypersurface to be a hyperplane of 1^{st} and 2^{nd} kind. The notations and terminologies of the present paper are referred from M. Matsumoto [5].

2. **Preliminaries**

Let us consider $F^n = (M^n, L(\alpha, \beta))$ with the following generalized (α, β)-metric [12, 13]

$$L(\alpha, \beta) = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}, \tag{2.1}$$

where $\mu_1, \mu_2\, and\, \mu_3$ are constants and $\mu_1 \neq 0$.

Differentiate partially (2.1) with respect to α and β, we get

$$\left.\begin{aligned} L_\alpha &= \frac{\mu_1\alpha^2 - \mu_3\beta^2}{\alpha^2}, \\ L_\beta &= \frac{\mu_2\alpha + 2\mu_3\beta}{\alpha}, \\ L_{\alpha\alpha} &= 2\mu_3\frac{\beta^2}{\alpha^3}, \\ L_{\beta\beta} &= \frac{2\mu_3}{\alpha}, \\ L_{\alpha\beta} &= -2\mu_3\frac{\beta}{\alpha^2},. \end{aligned}\right\} \tag{2.2}$$

where $L_\alpha = \frac{\partial L}{\partial\alpha},\ L_\beta = \frac{\partial L}{\partial\beta},\ L_{\alpha\alpha} = \frac{\partial^2 L}{\partial\alpha^2},\ L_{\beta\beta} = \frac{\partial^2 L}{\partial\beta^2},\ L_{\alpha\beta} = \frac{\partial^2 L}{\partial\alpha\partial\beta}$.

The normalized element of support $l_i = \partial_i L$ and h_{ij} of the special Finsler space $F^n = (M^n, L)$ are given by [17]

$$l_i = \alpha^{-1}L_\alpha Y^i + L_\beta b_i, \tag{2.3}$$

$$h_{ij} = pa_{ij} + q_0 b_i b_j + q_1(b_i Y_j + b_j Y_i) + q_2 Y_i Y_j, \tag{2.4}$$

where

$$\left.\begin{aligned} Y_i &= a_{ij}y^j, \\ p &= LL_\alpha\alpha^{-1}, \\ q_0 &= LL_{\beta\beta}, \\ q_1 &= LL_{\alpha\beta}\alpha^{-1}, \\ q_2 &= L\alpha^{-2}(L_{\alpha\alpha} - L_\alpha\alpha^{-1}). \end{aligned}\right\} \tag{2.5}$$

The fundamental tensor $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$ is given by [17]

$$g_{ij} = pa_{ij} + p_0 b_i b_j + p_1(b_i Y_j + b_j Y_i) + p_2 Y_i Y_j, \tag{2.6}$$

where

$$p_0 = q_0 + L_\beta^2,\ p_1 = q_1 + L^{-1}pL_\beta,\ p_2 = q_2 + p^2 L^{-2}. \tag{2.7}$$

Moreover, the reciprocal tensor g^{ij} of g_{ij} is given by

$$g^{ij} = p^{-1}a^{ij} - s_0 b^i b^j - s_1(b^i y^j + b^j y^i) - s_2 y^i y^j, \tag{2.8}$$

where

$$\begin{aligned} b^i &= a^{ij}b_j, \\ s_0 &= \{pp_0 + (p_0p_2 - p_1^2)\alpha^2\}/\zeta p, \\ s_1 &= \{pp_1 - (p_0p_2 - p_1^2)\beta\}/\zeta p, \\ s_2 &= \{pp_2 + (p_0p_2 - p_1^2)b^2\}/\zeta p, \;\; b^2 = a_{ij}b^ib^j, \\ \zeta &= p(p + p_0b^2 + p_1\beta) + (p_0p_2 - p_1^2)(\alpha^2b^2 - \beta^2). \end{aligned} \tag{2.9}$$

The Cartan tensor C_{ijk} is given by

$$2pC_{ijk} = p_1(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_i m_j m_k, \tag{2.10}$$

where

$$\gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3p_1q_0, \;\; m_i = b_i - \alpha^{-2}\beta y^i. \tag{2.11}$$

Let the components of Christoffel symbols of the associated Riemannian space R^n be $\left\{ \begin{array}{c} i \\ jk \end{array} \right\}$ and the covariant differentiation with respect to x^k relative to this Christoffel symbols be Δ_k.

We have the following tensors

$$\left.\begin{aligned} 2E_{ij} = b_{ij} + b_{ji}, \\ 2F_{ij} = b_{ij} - b_{ji}, \end{aligned}\right\} \tag{2.12}$$

where $b_{ij} = \Delta_j b_i$.

The difference tensor $D^i_{jk} = F^i_{jk} - \left\{ \begin{array}{c} i \\ jk \end{array} \right\}$ is given by

$$\begin{aligned} D^i_{jk} &= B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_j b_{0k} + B^i_k b_{0j} \\ &- b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} \\ &+ \lambda^s (C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}), \end{aligned} \tag{2.13}$$

where

$$\left.\begin{aligned} B_k = p_0 b_k + p_1 y_k, \quad B^i = g^{ij} B_j, \quad F^k_i = g^{kj} F_{ji}, \\ B_{ij} = \{p_1(a_{ij} - \alpha^{-2} y_i y_j) + \frac{\partial p_0}{\partial \beta} m_i m_j\}/2, \\ A^m_k = B^m_k E_{00} + B^m E_{k0} + B_k F^m_0 + B_0 F^m_k, \\ B^k_i = g^{kj} B_{ji}, \quad \lambda^m = B^m E_{00} + 2 B_0 F^m_0. \end{aligned}\right\} \tag{2.14}$$

Suffix $'0'$ denotes the contraction by the supporting element y^i except for the quantities $p_0, \ q_0 \ and \ s_0$.

3. Induced Cartan Connection

A hypersurface M^{n-1} of the M^n represented by the equation $x^i = x^i(u^\alpha), \alpha = 1, 2, ..., n-1$, where u^α are Gaussian coordinates on the hypersurface M^{n-1}. We assume that the matrix of projection factors $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$ is of rank $n-1$. If the supporting element y^i at a point $u = (u^\alpha)$ of M^{n-1} is assumed to be tangent to M^{n-1}, then $y^i = B^i_\alpha(u) v^\alpha$ such that $v = (v^\alpha)$ is thought of as the supporting element of M^{n-1} at the point u^α. We get $(n-1)$-dimensional Finsler space $F^{n-1} = (M^{n-1}, \underline{L}(u, v))$ because $\underline{L}(u, v) = L(x(u), y(u, v))$ gives rise to a Finslerian metric on M^{n-1},

The metric and Carton tensors $g_{\alpha\beta}$ and $C_{\alpha\beta\gamma}$ respectively are given by

$$\left.\begin{aligned} g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta, \\ C_{\alpha\beta\gamma} = C_{ijk} B^i_\alpha B^j_\beta B^k_\gamma. \end{aligned}\right\} \tag{3.1}$$

The unit normal $N^i(u,v)$ at each point u^α of F^{n-1} is defined by

$$\left.\begin{aligned} g_{ij}B^i_\alpha N^j &= 0,\\ g_{ij}N^iN^j &= 1.\end{aligned}\right\} \tag{3.2}$$

For the angular metric tensor h_{ij}, we have

$$\left.\begin{aligned} h_{\alpha\beta} &= h_{ij}B^i_\alpha B^j_\beta,\\ h_{ij}B^i_\alpha N^j &= 0,\\ h_{ij}N^iN^j &= 1.\end{aligned}\right\} \tag{3.3}$$

The inverse projection factor $B^\alpha_i(u,v)$ of B^i_α is defined as

$$B^\alpha_i = g^{\alpha\beta}g_{ij}B^j_\beta, \tag{3.4}$$

where $g^{\alpha\beta}$ is the inverse of $g_{\alpha\beta}$ of F^{n-1}.

From(3.2) and (3.4), we have

$$\left.\begin{aligned} B^i_\alpha B^\beta_i &= \delta^\beta_\alpha,\\ B^i_\alpha N_i &= 0,\\ N^iB^\alpha_i &= 0,\\ N^iN_i &= 1.\end{aligned}\right\} \tag{3.5}$$

Further, we have

$$B^i_\alpha B^\alpha_j + N^iN_j = \delta^i_j. \tag{3.6}$$

The second fundamental h-tensor $H_{\alpha\beta}$ and the normal curvature vector H_α of F^{n-1} are given by

$$H_{\alpha\beta} = N_i(B^i_{\alpha\beta} + F^i_{jk}B^j_\alpha B^k_\beta) + M_\alpha H_\beta \tag{3.7}$$

and

$$H_\alpha = N_i(B^i_{0\alpha} + G^i_jB^j_\alpha), \tag{3.8}$$

where $M_\alpha = C_{ijk} B^i_\alpha N^j N^k$, $B^i_{\alpha\beta} = \frac{\partial x^i}{\partial u^\alpha \partial u^\beta}$ and $B^i_{0\alpha} = B^i_{\beta\alpha} v^\beta$. It is clear that $H_{\alpha\beta}$ is not symmetric and

$$H_{\alpha\beta} - H_{\beta\alpha} = M_\alpha H_\beta - M_\beta H_\alpha. \tag{3.9}$$

The equations (3.7) and (3.8) yields

$$\left.\begin{aligned} H_{0\alpha} = H_{\beta\alpha} v^\beta = H_\alpha, \\ H_{\alpha 0} = H_{\alpha\beta} v^\beta = H_\alpha + M_\alpha H_0. \end{aligned}\right\} \tag{3.10}$$

The $M_{\alpha\beta}$ is given by

$$M_{\alpha\beta} = C_{ijk} B^i_\alpha B^j_\beta N^k. \tag{3.11}$$

The derivatives of B^i_α and N^i are as follows

$$\left.\begin{aligned} B^i_{\alpha|\beta} = H_{\alpha\beta} N^i, \\ B^i_{\alpha|\beta} = M_{\alpha\beta} N^i, \\ N^i_{|\beta} = -H_{\alpha\beta} B^\alpha_j g^{ij}, \\ N^i_{|\beta} = -M_{\alpha\beta} B^\alpha_j g^{ij}. \end{aligned}\right\} \tag{3.12}$$

The h and v-covariant derivatives of vector field X_i are

$$\left.\begin{aligned} X_{i|\beta} = X_{i|j} B^j_\beta + X_{i|j} N^j H_\beta, \\ X_{i|\beta} = X_{i|j} B^j_\beta. \end{aligned}\right\} \tag{3.13}$$

In 1985, M. Matsumoto [6] given the following

Lemma 3.1. *A hypersurface F^{n-1} is a hyperplane of the 1^{st} kind iff $H_\alpha = 0$ or equivalently $H_0 = 0$.*

Lemma 3.2. *A hypersurface F^{n-1} is a hyperplane of the 2^{nd} kind iff $H_{\alpha\beta} = 0$.*

Lemma 3.3. *A hypersurface F^{n-1} is a hyperplane of the 3^{rd} kind iff $H_{\alpha\beta} = 0 = M_{\alpha\beta}$.*

4. Finslerian Hypersurface with Generalized Metric $L = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$

Let us consider $F^n = (M^n, L(\alpha,\beta))$ with the following generalized (α,β)-metric

$$L(\alpha,\beta) = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$$

where μ_1, μ_2, μ_3 are constants and $\mu_1 \neq 0$. From the parametric equation $x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$, we get $\partial_\alpha b(x(u)) = 0 = b_i B^i_\alpha$,

Along $F^{n-1}(c)$, we have

$$\left.\begin{aligned} b_i B^i_\alpha &= 0, \\ b_i y^i &= 0. \end{aligned}\right\} \tag{4.1}$$

The following is the induced metric $\underline{L}(u,v)$ of $F^{n-1}(c)$ which is a Riemannian metric

$$\underline{L}(u,v) = \sqrt{a_{\alpha\beta}v^\alpha v^\beta}, \quad a_{\alpha\beta} = a_{ij}B^i_\alpha B^j_\beta. \tag{4.2}$$

Along $F^{n-1}(c)$, from (2.5), (2.7) and (2.9), we have

$$\begin{aligned} &p = \mu_1^2, \quad q_0 = 2\mu_1\mu_3, \quad q_1 = 0, \quad q_2 = \frac{\mu_1^2}{\alpha^6}, \\ &p_0 = 2\mu_1\mu_3 + \mu_2^2, \quad p_1 = \frac{\mu_1\mu_2}{\alpha}, \quad p_2 = 0, \quad \zeta = \mu_1^3(\mu_1 + 2\mu_3 b^2), \\ &s_0 = \frac{2\mu_3}{\mu_1^2(\mu_1 + 2\mu_3 b^2)}, \quad s_1 = \frac{\mu_2}{\alpha\mu_1^2(\mu_1 + 2\mu_3 b^2)}, \\ &s_2 = -\frac{\mu_2^2 b^2}{\alpha^2\mu_1^3(\mu_1 + 2\mu_3 b^2)}. \end{aligned} \tag{4.3}$$

Then (2.8) become

$$g^{ij} = \frac{a_{ij}}{\mu_1^2} - \frac{2\mu_3}{\mu_1^2(\mu_1 + 2\mu_3 b^2)}b^i b^j - \frac{\mu_2}{\alpha\mu_1^2(\mu_1 + 2\mu_3 b^2)}(b^i y^j + b^j y^i) + \frac{\mu_2^2 b^2}{\alpha^2\mu_1^3(\mu_1 + 2\mu_3 b^2)}y^i y^j. \tag{4.4}$$

Using (4.1) in (4.4), we get

$$g^{ij}b_i b_j = \frac{b^2}{\mu_1(\mu_1 + 2\mu_3 b^2)},$$

which gives

$$b_i(x(u)) = \sqrt{\frac{b^2}{\mu_1(\mu_1 + 2\mu_3 b^2)}}N_i, \tag{4.5}$$

where b is the length of the vector b^i.

Using (4.4) and (4.5), we can write

$$b^i = a_{ij}b_j = \sqrt{b^2\mu_1^3(\mu_1 + 2\mu_3 b^2)}N^i + \mu_2 b^2\alpha^{-1}y^i. \tag{4.6}$$

Thus, we have

Theorem 4.1. *The induced metric on special Finsler hypersurface $F^{n-1}(c)$ is Riemannian metric given by (4.2) and the scalar function $b(x)$ are given by (4.5) and (4.6).*

Along F^n, h_{ij} and g_{ij} are given by

$$h_{ij} = \mu_1^2 a_{ij} + 2\mu_1\mu_3 b_i b_j - \mu_1^2\alpha^{-2}y_i y_j, \tag{4.7}$$

$$g_{ij} = \mu_1^2 a_{ij} + (2\mu_1\mu_3 + \mu_2^2)b_i b_j + \mu_1\mu_2\alpha^{-1}(b_i y_j + b_j y_i). \tag{4.8}$$

Let a_{ij} be $h^{(a)}_{\alpha\beta}$.

Then using (4.1), (4.7) and (3.3), we can write

$$h_{\alpha\beta} = h^{(a)}_{\alpha\beta}.$$

From (2.7), we get

$$\frac{\partial p_0}{\partial \beta} = \frac{1}{\alpha^2}[6\mu_2\mu_3\alpha + 12\mu_3^2\beta].$$

Hence along $F^{n-1}(c)$, $\frac{\partial p_0}{\partial \beta} = \frac{6\mu_2\mu_3}{\alpha}$ and therefore (2.11) gives $\gamma_1 = 0, \quad m_i = b_i$.

Then Carton tensor becomes

$$Cijk = \frac{\mu_2}{2\mu_1\alpha}[h_{ij}m_k + h_{jk}m_i + h_{ki}m_j]. \tag{4.9}$$

Therefore using (3.3), (3.11) and (4.1), we get

$$M_{\alpha\beta} = \frac{\mu_2}{2\alpha}\sqrt{\frac{b^2}{\mu_1^3(\mu_1 + 2\mu_3 b^2)}}h_{\alpha\beta} \quad and \quad M_\alpha = 0 \tag{4.10}$$

and hence from (3.9) it follows that $H_{\alpha\beta}$ is symmetric.

Hence, we have

Theorem 4.2. *The 2^{nd} fundamental v-tensor of Finsler hypersurface F^{n-1} of Finsler space equipped with metric $L = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$, is given by (4.10) and the second fundamental h-tensor is symmetric.*

Taking h-covarient derivative of (4.1) with respect to the induced connection, we get

$$b_{i|\beta}B^i_\alpha + b_iB^i_{\alpha|\beta} = 0. \tag{4.11}$$

Applying (3.13) for the vector b_i, we get

$$b_{i|\beta} = b_{i|j}B^j_\beta + b_{i|j}N^jH_\beta.$$

Using this and $B^i_{\alpha|\beta} = H_{\alpha\beta}N^i$, (4.11) becomes

$$b_{i|j}B^i_\alpha B^j_\beta + b_{i|j}B^i_\alpha N^jH_\beta + b_iH_{\alpha\beta}N^i = 0. \tag{4.12}$$

Since $b_{i|j} = -b_hC^h_{ij}$, using (4.5) and (4.10), we get

$$b_{i|j}B^i_\alpha N^j = -\sqrt{\frac{b^2}{\mu_1(\mu_1 + 2\mu_3b^2)}}M_\alpha = 0.$$

Thus (4.12) become

$$\sqrt{\frac{b^2}{\mu_1(\mu_1 + 2\mu_3b^2)}}H_{\alpha\beta} + b_{i|j}B^i_\alpha B^j_\beta = 0. \tag{4.13}$$

It is clear that $b_{i|j}$ is symmetric because $H_{\alpha\beta}$ is symmetric.

Further contracting (4.13) with v^β and then with v^α, we get

$$\sqrt{\frac{b^2}{\mu_1(\mu_1 + 2\mu_3b^2)}}H_\alpha + b_{i|j}B^i_\alpha y^j = 0,$$

$$\sqrt{\frac{b^2}{\mu_1(\mu_1 + 2\mu_3b^2)}}H_0 + b_{i|j}y^iy^j = 0. \tag{4.14}$$

In view of lemma (3.1), the hypersurface $F^{n-1}(c)$ is a hyperplane of the 1^{st} kind if and only if $b_{i|j} = y^iy^j = 0$.

Here the covariant derivative $b_{i|j}$ depends on y^i.

From (2.12), we have $E_{ij} = b_{ij}, F_{ij} = 0$.

Thus (2.13) becomes

$$\begin{aligned} D^i_{jk} &= B^i b_{jk} + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} \\ &- C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} \\ &+ \lambda^s (C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}). \end{aligned} \tag{4.15}$$

In view of (4.3) and (4.4), the relations in (2.14) becomes

$$\begin{gathered} B_i = (2\mu_1\mu_3 + \mu_2^2) b_i + \tfrac{\mu_1\mu_2}{\alpha} y^i, \quad B^i = \tfrac{2\mu_3}{(\mu_1 + 2\mu_3 b^2)} b^i + \tfrac{\mu_2}{\alpha(\mu_1 + 2\mu_3 b^2)} y^i, \\ B_{ij} = \tfrac{1}{2\alpha} [\mu_1\mu_2 a_{ij} - \mu_1\mu_2 \alpha^{-2} y_i y_j + 6\mu_2\mu_3 b_i b_j], \quad B^i_j = g^{ik} B_{kj}, \\ A^m_k = B^m_k b_{00} + B^m b_{k0}, \quad \lambda^m = B^m b_{00}. \end{gathered} \tag{4.16}$$

By virtue of (4.1), we have $B_{i0} = 0, B^i_0 = 0$ which leads $A^m_0 = B^m b_{00}$.

Therefore we have

$$\begin{gathered} D^i_{j0} = B^i b_{j0} + B^i_j b_{00} - B^m C^i_{jm} b_{00}, \\ D^i_{00} = B^i b_{00} = [\tfrac{2\mu_3}{(\mu_1 + 2\mu_3 b^2)} b^i + \tfrac{\mu_2}{\alpha(\mu_1 + 2\mu_3 b^2)} y^i] b_{00}. \end{gathered} \tag{4.17}$$

Using the relation (4.1), we get

$$b_i D^i_{j0} = \frac{2\mu_3 b^2}{(\mu_1 + 2\mu_3 b^2)} b_{j0} + \frac{\mu_1\mu_2 + 6\mu_2\mu_3 b^2}{2\alpha\mu_1(\mu_1 + 2\mu_3 b^2)} b_j b_{00} - \frac{2\mu_3}{(\mu_1 + 2\mu_3 b^2)} b^m b_i C^i_{jm} b_{00}, \tag{4.18}$$

$$b_i D^i_{00} = \frac{2\mu_3 b^2}{(\mu_1 + 2\mu_3 b^2)} b_{00}. \tag{4.19}$$

Let $b_{i|j}$ denotes covariant derivative of b_i with respect to x^i relative to the Cartan connection of F^n and the covariant derivative of b_i with respect to x^j relative to the Riemannian connection is denoted by $b_{ij} = \Delta_j b_i$.

$$\begin{aligned} b_{i|j} - b_{ij} &= (\partial_j b_i - F^r_{ij} b_r) - (\partial_j b_i - \left\{ \begin{matrix} r \\ ij \end{matrix} \right\} b_r) \\ &= -(F^r_{ij} - \left\{ \begin{matrix} r \\ ij \end{matrix} \right\}) b_r = -D^r_{ij} b_r, \\ b_{i|j} &= b_{ij} - D^r_{ij} b_r. \end{aligned} \tag{4.20}$$

Using (4.19), we get

$$b_{i|j} y^i y^j = b_{00} - D^r_{00} b_r = \frac{\mu_1}{(\mu_1 + 2\mu_3 b^2)} b_{00}.$$

Consequently, (4.14) may be written as

$$\sqrt{\frac{b^2}{\mu_1(\mu_1 + 2\mu_3 b^2)}} H_0 + \frac{\mu_1}{(\mu_1 + 2\mu_3 b^2)} b_{00} = 0. \tag{4.21}$$

Thus the condition H_0 is equivalent to b_{00}, where b_{ij} does not depend upon y^i. Since y^i is to satisfy (4.1), the condition is written as $b_{ij} y^i y^j = (b_i y^i)(c_j y^j)$ for some $c_j(x)$, so that we have

$$2b_{ij} = b_i c_j + b_j c_i. \tag{4.22}$$

Using (4.1), it follows that

$$b_{00} = 0, \quad b_{ij} B^i_\alpha B^j_\beta = 0, \quad b_{ij} B^i_\alpha y^j = 0.$$

Again (4.16) and (4.22) gives

$$b_{i0} b^i = \frac{c_0 b^2}{2}, \quad \lambda^m = 0, \quad A^i_j B^j_\beta = 0, \quad B_{ij} B^i_\alpha B^j_\beta = \frac{1}{2\alpha} h_{\alpha\beta}.$$

Using the equations (3.11), (4.4), (4.6), (4.10) and (4.15), we can write

$$b_r D^r_{ij} B^i_\alpha B^j_\beta = -\frac{\mu_2 c_0 b^2}{4\alpha \mu_1^2 (\mu_1 + 2\mu_3 b^2)^2} h_{\alpha\beta}.$$

Therefore in view of (4.20), the equation (4.13) reduces to

$$\sqrt{\frac{b^2}{\mu_1(\mu_1 + 2\mu_3 b^2)}} H_{\alpha\beta} + \frac{\mu_2 c_0 b^2}{4\alpha \mu_1^2 (\mu_1 + 2\mu_3 b^2)^2} h_{\alpha\beta} = 0. \tag{4.23}$$

Theorem 4.3. *The necessary and sufficient condition for the hypersurface $F^{n-1}(c)$ of Finslerian space equipped with metric $L = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$, to be hyperplane of the 1^{st} kind is (4.22) and in this case the 2^{nd} fundamental h-tensor of hypersurface $F^{n-1}(c)$, is proportional to its angular metric tensor.*

By Lemma (3.2), the hypersurface $F^{n-1}(c)$ is a hyperplane of the 2^{nd} kind if and only if $H_{\alpha\beta} = 0$. Thus from (4.23) we get $c_0 = c_i(x)y^i = 0$. Therefore $\exists\ e(x) \ni c_i(x) = e(x)b_i(x)$. Hence (4.22) gives

$$b_{ij} = eb_ib_j. \tag{4.24}$$

Theorem 4.4. *The necessary and sufficient condition for the hypersurface $F^{n-1}(c)$ of Finslerian space equipped with metric $L = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$, to be hyperplane of the 2^{nd} kind is (4.24).*

In view of equation (4.10) and Lemma(3.3), we have

Theorem 4.5. *The hypersurface $F^{n-1}(c)$ of Finslerian space equipped with metric $L = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$ is not hyperplane of the 3^{rd} kind.*

5. Conclusion

In this paper, we have investigated various kinds of hypersurfaces of Finsler space with special $L(\alpha, \beta) = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$ (where μ_1, μ_2, μ_3 are constants and $\mu_1 \neq 0$). Also we have obtained conditions for these hypersurfaces to be a hyperplane of 1^{st} and 2^{nd} kind.

References

[1] M. K. Gupta and P. N. Pandey, *Hypersurface of conformally and h-conformally related Finsler spaces*, Acta Math. Hungar., 123(3)(2009), 257-264.

[2] M. K. Gupta and P. N. Pandey, *On hypersurface of a Finsler space with a special metric*, Acta Math. Hungar., 120(12)(2008), 165-177.

[3] M. Hashiguchi and Y. Ichijyo, *On some special* (α, β)*-metric,* Rep. Fac. Sci. Kagasima Univ. (Math., Phys., Chem.), 8(1975), 39-46.

[4] S. Kikuchi, *On the condition that a space with* (α, β)*-metric be locally Minkowskian,* Tensor, N.S., 33(1979), 242-246.

[5] M. Matsumoto, *Foundations of Finsler geometry and Finsler spaces*, Japan: Kaiseisha Press, 520(1986).

[6] M. Matsumoto, *The induced and intrinsic Finsler connections of a hypersurface and Finslerien projective geometry*, J. Math. Kyoto Univ., 25(1)1985, 107-144.

[7] M. Matsumoto,*Theory of Finsler spaces with* (α, β)*-metric,* Rep. Math. Phys., 30(1991), 15-20.

[8] S. K. Narasimhamurthy, Ajith, C. S. Bagewadi and Pradeep Kumar, *Finsler hypersurface given by generalized* β*-conformal change*, Journal of International Academy of Physical Sciences, 14(3)(2010), 273-282.

[9] S. K. Narasimhamurthy, S. T. Aveesh, H. G. Nagaraja and Pradeep Kumar, *On special hypersurface of a finsler space with the metric* $\alpha + \frac{\beta^{n+1}}{\alpha^n}$, Acta Universitatis Apulensis, 17(2009), 129-139.

[10] S. K. Narasimhamurthy, Pradeep Kumar and S. T. Aveesh, *A study of hypersurfaces of special Finsler space*, Int. Journal Pure & Applied Mathematics, 48(1)(2008), 67-74.

[11] S. K. Narasimhamurthy, Pradeep Kumar and C. S. Bagewadi, *C-conformal metric transformations on Finslerian hypersurface*, J. of Indonesian Mathematical society, 17(2)(2011), 59-66.

[12] Pradeep Kumar, Madhu T. S. and Chandru K., *Einstein Finslerian space with special* (α, β)*-metric*, Journal of Computer and Mathematical Sciences, 10(3)(2019), 392-398.

[13] Pradeep Kumar, Madhu T. S. and Sharath B. R., *Projective changes between generalized* (α, β)*-metric and Randers metric*, Advances in Pure Mathematics, 10(5)(2020), 312-321.

[14] Pradeep Kumar, S.K. Narasimhamurthy and C.S. Bagewadi, *A study of hypersurfaces of special Finsler space*, Int. e-Jour Engg. Mathematics: Theory and Application, 4(1)(2008), 131-139.

[15] Pradeep Kumar, S.K. Narasimhamurthy, C.S. Bagewadi and S.T. Aveesh, *Special Finsler hypersurfaces admitting a parallel vector field*, SCIENTIA, Series A: Mathematical Science, 20(2010), 123-130.

[16] Pradeep Kumar, S. K. Narasimhamurthy, H. G. Nagaraja and S. T. Aveesh, *On a special hypersurface of a Finsler space with* (α, β)*-metric*, Tbilisi Mathematical Journal, 2(2009), 51-60.

[17] G. Randers, *On an asymmetric metric in the four spaces of general relativity*, Phys. Rev., 59(1941), 195-199.

[18] C. Shibata, *On Finsler spaces whih an* (α, β)*-metric,* J. Hollaido Univ. of Education, IIA 35(1984), 1-16.

[19] I-Yong Lee, Ha-Yong park and Yong-Duk Lee, *On hypersurface of a special Finsler space with a metric* $L = \alpha + \frac{\beta^2}{\alpha}$, Korean J. Math. Sciences, 8(2001), 93-101.