η-Ricci soliton in three-dimensional (ϵ, δ)-trans-Sasakian manifold.

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Abstract: The objective of the present paper is to carry out η-Ricci soliton in three-dimensional (ε, δ)-trans-Sasakian manifold admitting K-torse forming vector field and we have got some geometrical results. The above mentioned team has studied Einstein like and (ε, δ)-trans-Sasakian manifold admitting η-Ricci soliton. Further we study the property of the M-projective curvature tensor in (ε, δ)-trans-Sasakian manifold. Finally we wind up by η-Ricci soliton in (ε, δ)-trans-Sasakian manifold on a view to parallel symmetric (0, 2)-tensor field.

Key Words: Ricci soliton, η-Ricci soliton,(ε, δ) trans-Sasakian manifold, K-torse forming vector field, M-projective curvature tensor.

1. Introduction

A. Bejancu and K. L. Duggal [1] initiated the notion of (ε)-"Sasakian manifolds" and the extended work on this was carried out by Xufend and Xiaoli [10] and Rakesh Kumar et. al. [9]. De and Sarkar [2] investigated which is conformally flat, Weyl semisymmetric, ϕ-recurrent (ε)-Kenmotsu manifolds. In [1], the researchers obtained Riemannian curvature tensor of (ε)-"Sasakian manifolds" and have established various relations of curvatures which are in different forms. H.G. Nagaraja et. al. [6] and Y.B. Maralabhavi et. al. [18] have further enquired with (ε, δ)-"trans-Sasakian structures" and generalised manifolds of both (ε) and (ε)-Kenmotsu.

In a Riemannian manifold (M, g), the metric g is called a "Ricci soliton" if [16]

$$(L_V g)(U_1, U_2) + 2S(U_1, U_2) + 2\lambda g(U_1, U_2) = 0., \tag{1.1}$$

where L, S and V are denotes the Lie-derivative, the Ricci tensor and a complete vector field on M respectively and λ is a constant. The "Ricci soliton" is said to be expanding, shrinking and steady if λ is positive, negative and zero respectively. Cho and Kimura [11] studied the notion of η-Ricci soliton $(\zeta, g, V, \lambda, \mu)$ is the generalisation of the "Ricci soliton" (ζ, g, V, λ) and is defined as

$$(L_V g)(U_1, U_2) + 2S(U_1, U_2) + 2\lambda g(U_1, U_2) + 2\mu\eta(U_1)\eta(U_2) = 0. \tag{1.2}$$

where μ is a constant.

We made an enquiry into the vector field of K-torse forming of the (ε, δ)-"trans-Sasakian manifold" which is of three dimension. S. Yamaguchi [15] initiated the notion of K-torse forming vector field

$$\nabla_{U_1}\rho = l_1 U_1 + l_2\phi U_1 + l_3(U_1)\rho + l_4(U_1)\phi\rho. \tag{1.3}$$

where l_1, l_2 are functions , $l_3(U_1)$, $l_4(U_1)$ are 1-forms and ρ is a vector field.

Assume (M, g) as three-dimensional differentiable manifold with the g metric and the ∇ "Riemannian connection" . The M-projective curvature tensor is defined as [17]

$$\begin{aligned} M(U_1, U_2)U_3 &= R(U_1, U_2)U_3 - \frac{1}{2(n-1)}[S(U_2, U_3)U_1 - S(U_1, U_3)U_2 + g(U_2, U_3)QU_1 \\ &\quad -g(U_1, U_3)QU_2], \end{aligned} \tag{1.4}$$

In this paper we study η-Ricci soliton in $[(\text{t-}Sm)_3]$.

Where $[(\text{t-}Sm)_3]$ stands for three-dimensional (ε, δ)-"trans-Sasakian manifold" and $[(\text{t-}Sm)]$ stands for (ε, δ)-"trans-Sasakian manifold"

2. preliminary Matter

Assume (M, g) be an "almost contact metric manifold" with odd dimension consisting of a g Riemannian metric, η a 1-form, ζ a vector field, (1, 1) tensor field forming (ϕ, ζ, η, g) contact metric structure satisfying

$$\phi^2 = -I + \eta \otimes \zeta, \tag{2.1}$$

$$\eta(\zeta) = 1, \tag{2.2}$$

$$\phi\zeta = 0, \eta \circ \phi = 0 \tag{2.3}$$

Manifold of almost contact metric M is said to be (ε)-"almost contact metric manifold" when

$$g(\zeta, \zeta) = \varepsilon, \tag{2.4}$$

$$\eta(U_1) = \varepsilon g(U_1, \zeta), \tag{2.5}$$

$$g(\phi U_1, \phi U_2) = g(U_1, U_2) - \varepsilon\eta(U_1)\eta(U_2), \ \ \forall U_1, U_2 \in TM, \tag{2.6}$$

Where $\varepsilon = g(\zeta, \zeta) = \pm 1$. Manifold of (ε)-"almost contact metric" can be called as [(t-Sm)] if

$$(\nabla_{U_1}\phi)U_2 = \alpha[g(U_1, U_2)\zeta - \varepsilon\eta(U_2)U_1] + \beta[g(\phi U_1, U_2)\zeta - \delta\eta(U_2)\phi U_1], \tag{2.7}$$

holds good for some smooth functions α and β on M and $\delta = \pm 1, \varepsilon = \pm 1$. For $\alpha = 1, \beta = 0$, an [(t-Sm)] gets reduced to an (ε)-Sasakian and for $\alpha = 0, \beta = 1$ it reduces to a manifold of (δ)-Kenmotsu.

Let (M, g) be a [(t-Sm)]. Then from (2.7), it can be convieniently seen that

$$\nabla_{U_1}\zeta = -\varepsilon\alpha\phi U_1 - \beta\delta\phi^2 U_1, \tag{2.8}$$

$$(\nabla_{U_1}\eta)U_2 = -\alpha g(U_2, \phi U_1) + \varepsilon\beta\delta g(\phi U_1, \phi U_2), \tag{2.9}$$

$$\zeta\alpha + 2\alpha\beta = 0. \tag{2.10}$$

In a [(t-$Sm)_3$], R is the curvature tensor and S is Ricci tensor will be represented by [4]

$$\begin{aligned} R(U_1,U_2)U_3 &= (2A-\frac{r}{2})[g(U_2,U_3)U_1-g(U_1,U_3)U_2]+B[(g(U_2,U_3)\eta(U_1)\zeta \\ &\quad -g(U_1,U_3)\eta(U_2)\zeta)+\eta(U_3)(\eta(U_2)U_1-\eta(U_1)U_2)], \end{aligned} \tag{2.11}$$

3. An η-Ricci soliton on Einstein-like [(t-Sm)]

Definition 3.1. *An [(t-Sm)] $(M,\phi,\zeta,\eta,g,\varepsilon)$ is known as Einstein-like if S Ricci tensor satisfies the following*

$$S(U_1,U_2)=a_1g(U_1,U_2)+a_2g(\phi U_1,U_2)+a_3\eta(U_1)\eta(U_2),\quad U_1,U_2\in\Gamma(TM) \tag{3.1}$$

for some real constant a_1,a_2,a_3.
Putting $V=\zeta$ and using (3.1) and (1.2), we have

$$g(\nabla_{U_1}\zeta,U_2)+g(U_1,\nabla_{U_2}\zeta)+2(a_1+\lambda)g(U_1,U_2)+2(a_3+\mu)\eta(U_1)\eta(U_2)+2a_2g(\phi U_1,U_2)=0 \tag{3.2}$$

In [(t-$Sm)_3$], ζ is always a K-torse forming vector field. Take $\rho=\zeta$ in (1.3) and comparing with (2.8) to get [18]

$$\nabla_{U_1}\zeta=\beta\delta U_1-\varepsilon\alpha\phi U_1-\beta\delta\eta(U_1)\zeta. \tag{3.3}$$

Using (3.3) in (3.2), we get

$$\beta\delta g(U_1,U_2)+(a_3+\mu)\eta(U_1)\eta(U_2)+a_2g(\phi U_1,U_2)=\varepsilon\beta\delta\eta(U_1)\eta(U_2)-(a_1+\lambda)g(U_1,U_2), \tag{3.4}$$

in view of (3.3) and (3.4), we obtain

$$\nabla_{U_1}\zeta=(a_1+\lambda)(-U_1+\eta(U_1)\zeta)-\varepsilon\alpha\phi U_1. \tag{3.5}$$

In consequence of (3.3), (3.5), we have from (3.1),

$$S(U_1,U_2)=-(\lambda+\beta\delta)g(U_1,U_2)+(\varepsilon\beta\delta-\mu)\eta(U_1)\eta(U_2), \tag{3.6}$$

$$S(U_1,\zeta)=-(\varepsilon\lambda+\mu)\eta(U_1). \tag{3.7}$$

comparing (3.6) with (3.1), we have

$a_1 = -(\lambda + \beta\delta), \;\; a_2 = 0 \;\; and \;\; a_3 = (\varepsilon\beta\delta - \mu)$

thus we can state the following

Theorem 3.1. *If ζ is a K-torse-forming "Ricci soliton" on an Einstein like Ricci tensor in [(t-Sm)] (M, g, ζ, λ), then the "Ricci soliton" will shrink, expand and steady accordingly as $a_1 + \beta\delta > 0$, $a_1 + \beta\delta < 0$ and $a_1 + \beta\delta = 0$.*

Above theorem leads to the following corollary

Corollary 3.1. *If ζ is a K-torse-forming "Ricci soliton" on an Einstein like Ricci tensor in [(t-Sm)] (M, g, ζ, λ) reduces to an ε-Sasakian for $\beta = 0$, then the "Ricci soliton" will shrink, expand and steady accordingly as $a_1 > 0$, $a_1 < 0$ and $a_1 = 0$.*

From (3.1), we have

$$(\nabla_W S)(U_2, U_3) = a_2 g((\nabla_W \phi)U_2, U_3) + a_3[\eta(U_3)g(\nabla_W \zeta, U_2) + \eta(U_2)g(\nabla_W \zeta, U_3)] \quad (3.8)$$

From (3.8), we obtain

$$(\nabla_W Q)U_2 = a_2(\nabla_W \phi)U_2 + a_3[g(\nabla_W \zeta, U_2)\zeta + \eta(U_2)\nabla_W \zeta] \quad (3.9)$$

where Q is the Ricci operator defined by$g(QU_1, U_2) = S(U_1, U_2), \;\; U_1, U_2 \in \Gamma(TM)$.
Furthermore, if the manifold is (ε, δ)-trans-Sasakian, then the scalar curvature is given by

$$r = 3\lambda + \varepsilon\mu, \quad (3.10)$$

In Einstein-like [t-Sm], the Ricci operator Q is of the form

$$Q = \lambda I + \varepsilon\mu\eta \otimes \zeta \quad (3.11)$$

for the Eigen value $\lambda + \varepsilon\mu$, the structure vector field ζ is an Eigen vector of Q.
If we take $U_1 = U_2 = \zeta$ in (3.2), we obtain

$$\varepsilon(a_1 + \lambda) + a_3 + \mu = 0, \quad (3.12)$$

Using (3.12) and taking $U_2 = \zeta$ in (3.2), we get

$$g(\nabla_\zeta \zeta, U_1) = 0 \quad (3.13)$$

which implies $\nabla_\zeta \zeta = 0$. So it is easy to see that

$$(\nabla_\zeta \phi)\zeta = 0 \ \ and \ \ \nabla_\zeta \eta = 0. \tag{3.14}$$

Take $W = \zeta$ in (3.8), (3.9) and by virtue of (3.14), we have

$$(\nabla_\zeta S)(U_2, U_3) = a_2 g((\nabla_\zeta \phi)U_2, U_3), \tag{3.15}$$

and

$$\nabla_\zeta Q = a_2 \nabla_\zeta \phi. \tag{3.16}$$

This leads the following

Proposition 1. *Assume $(M, \phi, \zeta, \eta, g, \varepsilon, a_1, a_2, a_3)$ be an Einstein-like [t-Sm] admitting an η-"Ricci soliton" (g, ζ, λ, μ). Then*
(a) $\varepsilon(a_1 + \lambda) + a_3 + \mu = 0$,
(b) ζ is a geodesic vector field,
(c) $(\nabla_\zeta \phi)\zeta = 0$ and $\nabla_\zeta \eta = 0$,
(d) $(\nabla_\zeta S)(U_2, U_3) = a_2 g((\nabla_\zeta \phi)U_2, U_3)$ and $\nabla_\zeta Q = a_2 \nabla_\zeta \phi$.
Furthermore, if $\nabla_\zeta S = 0$ and $\nabla_\zeta Q = 0$, then the manifold is (ε, δ)-trans-Sasakian.

Using (3.5), we have

$$\begin{aligned} R(U_1, U_2)\zeta &= 2\varepsilon\alpha(a_1 + \lambda)[\eta(U_1)\phi U_2 - \eta(U_2)\phi U_1] \\ &\quad + [(a_1 + \lambda)^2 - \alpha][\eta(U_1)U_2 - \eta(U_2)U_1], \end{aligned} \tag{3.17}$$

$$\begin{aligned} R(\zeta, U_1)U_2 &= -2\varepsilon\alpha(a_1 + \lambda)[\eta(U_2)\phi U_1 + g(\phi U_1, U_2)\zeta] \\ &\quad + [(a_1 + \lambda)^2 - \alpha][\eta(U_2)U_1 - g(U_1, U_2)\zeta], \end{aligned} \tag{3.18}$$

$$\begin{aligned} R(U_1, U_2)W &= 2\varepsilon\alpha(a_1 + \lambda)[g(U_2, \phi W)U_1 + g(\phi U_1, W)U_2] \\ &\quad - [(a_1 + \lambda)^2 - \alpha][g(U_2, W)U_1 - g(U_1, W)U_2], \end{aligned} \tag{3.19}$$

$$\begin{aligned} \eta(R(U_1, U_2)W) &= 2\varepsilon\alpha(a_1 + \lambda)[g(U_2, \phi W)\eta(U_1) + g(\phi U_1, W)\eta(U_2)] \\ &\quad - \varepsilon[(a_1 + \lambda)^2 - \alpha][g(U_2, W)\eta(U_1) - g(U_1, W)\eta(U_2)]. \end{aligned} \tag{3.20}$$

A [(t-Sm)$_3$] (M, g) is known as generalised recurrent [12], provided that R the curvature tensor satisfies the bellow condition

$$(\nabla_{U_1}R)(U_2,U_3)(W)=K(U_1)R(U_2,U_3)W+L(U_1)[g(U_3,W)U_2-g(U_2,W)U_3], \quad (3.21)$$

where K and L are 1-forms and $L\neq 0$.
Taking $U_2=W=\zeta$

$$(\nabla_{U_1}R)(\zeta,U_3)(\zeta)=K(U_1)R(\zeta,U_3)\zeta+L(U_1)[\eta(U_3)\zeta-U_3], \quad (3.22)$$

from covariant derivative's definition, we have

$$(\nabla_{U_1}R)(\zeta,U_3)(\zeta)=\nabla_{U_1}R(\zeta,U_3)\zeta-R(\nabla_{U_1}\zeta,U_3)\zeta-R(\zeta,\nabla_{U_1}U_3)\zeta-R(\zeta,U_3)\nabla_{U_1}\zeta. \quad (3.23)$$

by virtue of (3.5), (3.17) and (3.18), (3.23) implies

$$\begin{aligned}(\nabla_{U_1}R)(\zeta,U_3)(\zeta) \;=\;& d[2\varepsilon\alpha(a_1+\lambda)](U_1)(-\phi U_3)+d[(a_1+\lambda)^2-\alpha](U_1)[U_3-\eta(U_3)\zeta]\\ &-2\varepsilon\alpha(a_1+\lambda)[(a_1+\lambda)\eta(U_3)\phi U_1-\varepsilon\alpha\eta(U_3)U_1+\varepsilon\alpha\eta(U_1)\eta(U_3)\zeta]\\ &-[(a_1+\lambda)^2-\alpha][(a_1+\lambda)(\eta(U_3)U_1-\eta(U_1)\eta(U_3)\zeta)+\varepsilon\alpha\eta(U_3)\phi U_1]\\ &-2\varepsilon\alpha(a_1+\lambda)\phi\nabla_{U_1}U_3-[(a_1+\lambda)^2-\alpha][\nabla_{U_1}U_3-\eta(\nabla_{U_1}U_3)\zeta]\\ &-[(a_1+\lambda)^2-\alpha](a_1+\lambda)[2\eta(U_1)U_3-\eta(U_1)\eta(U_3)\zeta-g(U_1,U_3)\zeta]\\ &-2\varepsilon\alpha(a_1+\lambda)^2g(\phi U_1,U_3)\zeta+2\alpha^2(a_1+\lambda)g(U_1,U_3)\zeta\\ &-2\alpha^2(a_1+\lambda)\eta(U_1)\eta(U_3)\zeta-[(a_1+\lambda)^2-\alpha]\varepsilon\alpha g(\phi U_1,U_3)\zeta. \quad (3.24)\end{aligned}$$

Put $U_3=\zeta$ and use (2.2), (2.4) and (2.5) in (3.24) to get

$$[(a_1+\lambda)^2-\alpha]\eta(\nabla_{U_1}\zeta)\zeta=0, \quad (3.25)$$

if $[(a_1+\lambda)^2-\alpha]\neq 0$, then

$$\nabla_{U_1}\zeta=0. \quad (3.26)$$

Thus we have

Theorem 3.2. *In a generalised recurrent [(t-Sm)$_3$] M admitting η-"Ricci soliton" with the vector field ζ is co-symplectic provided $[(a_1+\lambda)^2-\alpha]\neq 0$.*

Above theorem leads to the following corollary

Proposition 2. *In a generalised recurrent [(t-Sm)$_3$] M admitting η-"Ricci soliton" with the vector field ζ will*
(a) expand if $\sqrt{\alpha} > a_1$
(b) shrink if $\sqrt{\alpha} < a_1$
(c) steady if $\sqrt{\alpha} = a_1$
provided $\nabla_{X_1}\zeta \neq 0$.

4. η- Ricci solitons in a [(t-Sm)$_3$]

Let M be an [(t-Sm)$_3$] admitting a "Ricci soliton" (g, V, λ). A "Ricci soliton" is a generalisation of an Einstein metric and defined based on a "Riemannian manifold" (M, g) by (1.2).
Consider V be a point wise collinear with ζ ie., $V = \gamma\zeta$, such that (1.2) becomes

$$g(\nabla_{U_1}\gamma\zeta, U_2) + g(\nabla_{U_2}\gamma\zeta, U_1) + 2S(U_1, U_2) + 2\lambda g(U_1, U_2) + 2\mu\eta(U_1)\eta(U_2) = 0, \tag{4.1}$$

which is reduced to

$$\gamma g(\nabla_{U_1}\zeta, U_2) + (U_1\gamma)g(U_2, \zeta) + \gamma g(\nabla_{U_2}\zeta, U_1) + (U_2\gamma)g(U_1, \zeta) \\ +2S(U_1, U_2) + 2\lambda g(U_1, U_2) + 2\mu\eta(U_1)\eta(U_2) = 0. \tag{4.2}$$

Taking $U_2 = \zeta$ and using (3.5), equation (4.2) can be written as

$$\eta(U_1)[2\varepsilon\lambda + 2\mu + \varepsilon\zeta\gamma + 2\varepsilon a_1 + 2a_3] + (U_1\gamma)\varepsilon = 0, \tag{4.3}$$

using $U_1 = \zeta$

$$\zeta\gamma = -(\lambda + a_1 + \varepsilon(\mu + a_3)), \tag{4.4}$$

by using (4.4) in (4.3), we get

$$U_1\gamma = -[\lambda + a_1 + \varepsilon(\mu + a_3)]\eta(U_1), \tag{4.5}$$

or

$$d\gamma = -[\lambda + a_1 + \varepsilon(\mu + a_3)]\eta, \tag{4.6}$$

consider $\nabla_{U_1} = d$.
Operating d on (4.6), we have

$$[\lambda + a_1 + \varepsilon(\mu + a_3)]d\eta = 0, \tag{4.7}$$

as $d\eta \neq 0$, we obtain

$$\lambda + \varepsilon\mu = -(a_1 + \varepsilon a_3), \tag{4.8}$$

using (4.8) in (4.6) implies $d\gamma = 0$. Hence γ is constant.
$\therefore$ From (4.2), we have

$$S(U_1, U_2) = b_1 g(U_1, U_2) + b_2 \eta(U_1)\eta(U_2), \tag{4.9}$$

where $b_1 = \gamma(a_1 + \lambda) - \lambda$ and $b_2 = -[\varepsilon\gamma(a_1 + \lambda) + \mu]$,
from (4.9), we have

$$r = 2\gamma(a_1 + \lambda) - (\lambda + \varepsilon\mu) \tag{4.10}$$

which leads the bellow result

Theorem 4.3. *Let M be a $[(t\text{-}Sm)_3]$ on an η-"Ricci soliton" along with the vector field V as point wise collinear with ζ. Then following results holds good*
(a) $\lambda + \varepsilon\mu = -(a_1 + \varepsilon a_3)$,
(b) M be an η-Einstein,
(c) M is of constant scalar curvature.

Suppose $R(\zeta, .).S = 0$.

$$S(R(\zeta, U_1)U_2, U_3) + S(U_2, R(\zeta, U_1)U_3) = 0, \quad \forall U_1, U_2, U_3 \in \Gamma(TM). \tag{4.11}$$

By virtue of (1.2) , (3.7) and (3.18)

$$\begin{aligned}
&2\varepsilon\alpha(a_1 + \lambda)[(\lambda + \beta\delta)(\eta(U_2)g(\phi U_1, U_3) + \eta(U_3)g(\phi U_1, U_2)) \\
&-(\varepsilon\lambda + \mu)(\eta(U_2)\eta(U_3))(g(\phi U_1, U_2) + g(\phi U_1, U_3)] \\
&+[(a_1 + \lambda)^2 - \alpha][-(\lambda + \mu)\eta(U_2)g(U_1, U_3) + 2(\varepsilon\beta\delta - \mu)\eta(U_1)\eta(U_2)\eta(U_3) \\
&+(\varepsilon\lambda + \mu)(\eta(U_3)g(U_1, U_2) + \eta(U_2)g(U_1, U_3)) \\
&-(\lambda + \beta\delta)\eta(U_3)g(U_1, U_2)] = 0.
\end{aligned} \tag{4.12}$$

Taking $U_1 = U_2 = U_3 = \zeta$, we get

$$2\lambda - \mu + \frac{a_3}{1-\varepsilon} = 0, \tag{4.13}$$

thus it can be shown as

Corollary 4.2. *On [(t-Sm)$_3$] admitting an η-"Ricci soliton" (g, ζ, λ, μ) along with K-torse forming vector field, we obtain*
$\lambda - \frac{\mu}{2} = \frac{a_3}{2(\varepsilon-1)}$.

Corollary 4.3. *On [(t-Sm)$_3$] admitting a "Ricci soliton" (g, ζ, λ) along with K-torse forming vector field.*
$\lambda = \frac{a_3}{2(\varepsilon-1)}$,
therefore "Ricci soliton" will
(a) expand if $\frac{a_3}{2(\varepsilon-1)} > 0$,
(b) shrink if $\frac{a_3}{2(\varepsilon-1)} < 0$,
(c) steady if $a_3 = 0$
provided V is a constant multiple of ζ.

5. [(t-Sm)] satisfying $R(U_1, U_2).M_p = 0$

In view of (3.6), equation (1.4) implies

$$\begin{aligned} M_p(U_1,U_2)U_3 &= R(U_1,U_2)U_3 - \frac{1}{4}[2(\lambda+\beta\delta)(g(U_1,U_3)U_2 - g(U_2,U_3)U_1) \\ &\quad +(\varepsilon\beta\delta-\mu)(\eta(U_2)\eta(U_3)U_1 - \eta(U_1)\eta(U_3)U_2 \\ &\quad +g(U_2,U_3)\eta(U_1)\zeta - g(U_1,U_3)\eta(U_2)\zeta)]. \end{aligned} \tag{5.1}$$

Using (2.2), (2.5), (3.20) and (5.1), we obtain

$$\begin{aligned} \eta(M_p(U_1,U_2)U_3) &= -2\alpha\beta\delta[\eta(U_1)g(U_2,\phi U_3) + \eta(U_2)g(\phi U_1,U_3)] \\ &\quad +[\frac{2\varepsilon(\lambda+\beta\delta)-(\beta\delta-\mu)}{4} - \\ &\quad \varepsilon(\beta^2-\alpha)][\eta(U_1)g(U_2,U_3) + \eta(U_2)g(U_1,U_3)], \end{aligned} \tag{5.2}$$

putting $U_3 = \zeta$ in (5.2), we get

$$\eta(M_p(U_1, U_2)\zeta) = 0. \tag{5.3}$$

Now

$$\begin{aligned} R(U_1, U_2).M_p(U_3, Z_1)Z_2 &= R(U_1, U_2)M_p(U_3, Z_1)Z_2 - M_p(R(U_1, U_2)U_3, Z_1)Z_2 - \\ &\quad M_p(U_3, R(U_1, U_2)Z_1)Z_2 - M_p(U_3, Z_1)R(U_1, U_2)Z_2. \end{aligned} \tag{5.4}$$

Using $R(U_1, U_2).M_p = 0$ in (5.4) we get

$$\begin{aligned} &R(U_1, U_2)M_p(U_3, Z_1)Z_2 - M_p(R(U_1, U_2)U_3, Z_1)_2 - \\ &M_p(U_3, R(U_1, U_2)Z_1)Z_2 - M_p(U_3, Z_1)R(U_1, U_2)Z_2 = 0. \end{aligned} \tag{5.5}$$

In view of (2.2)and (2.5), (5.5) becomes

$$\begin{aligned} &g(R(U_1, U_2)M_p(U_3, Z_1)Z_2, \zeta) - g(M_p(R(U_1, U_2)U_3, Z_1)Z_2, \zeta) - \\ &g(M_p(U_3, R(U_1, U_2)Z_1)Z_2, \zeta) - g(M_p(U_3, Z_1)R(U_1, U_2)Z_2, \zeta) = 0. \end{aligned} \tag{5.6}$$

Taking $U_1 = \zeta$ and by virtue of (2.2), (2.5), (3.18) and (5.2), we obtain

$$\begin{aligned} g(M_p(U_3, Z_1)Z_2, U_2) &= 2\varepsilon\beta\delta[g(Z_1, \phi Z_2)g(U_2, U_3) + g(\phi U_3, Z_2)g(Z_1, U_2) \\ &\quad +\eta(U_3)\eta(Z_2)g(\phi Z_1, U_2) - \eta(Z_1)\eta(Z_2)g(U_2, \phi U_3)] \\ &\quad +[\frac{2\varepsilon(\lambda + \beta\delta) - (\beta\delta - \varepsilon\mu)}{4} - \varepsilon(\beta^2 - \alpha)] \\ &\quad [g(U_3, Z_2)g(Z_1, U_2) - g(Z_1, Z_2)g(U_2, U_3)], \end{aligned} \tag{5.7}$$

$$\begin{aligned} M_p(U_3, Z_1)Z_2 &= 2\varepsilon\beta\delta[g(Z_1, \phi Z_2)U_3 + g(\phi U_3, Z_2)Z_1 + \eta(U_3)\eta(Z_2)\phi Z_1 \\ &\quad -\eta(Z_1)\eta(Z_2)\phi U_3] + [\frac{2\varepsilon(\lambda + \beta\delta) - (\beta\delta - \varepsilon\mu)}{4} \\ &\quad -\varepsilon(\beta^2 - \alpha)][g(U_3, Z_2)Z_1 - g(Z_1, Z_2)U_3]. \end{aligned} \tag{5.8}$$

By virtue of (5.1) and (5.8)

$$\begin{aligned} R(U_3, Z_1)Z_2 &= 2\varepsilon\beta\delta[g(Z_1, \phi Z_2)U_3 + g(\phi U_3, Z_2)Z_1 + \eta(U_3)\eta(Z_2)\phi Z_1 - \eta(Z_1)\eta(Z_2)\phi U_3] \\ &+ \frac{(\varepsilon\beta\delta - \mu)}{4}[g(Z_1, Z_2)\eta(U_3)\zeta - g(U_3, Z_2)\eta(Z_1)\zeta + \eta(Z_1)\eta(Z_2)U_3 - \eta(U_3)\eta(Z_2)Z_1] \\ &[\frac{2(\varepsilon+1)(\lambda+\beta\delta) - (\beta\delta - \varepsilon\mu)}{4} - \varepsilon(\beta^2 - \alpha)][g(U_3, Z_2)Z_1 - g(Z_1, Z_2)U_3]. \end{aligned} \tag{5.9}$$

Contracting (5.9) with respect to W, we get

$$S(Z_1, Z_2) = ag(Z_1, Z_2) + bg(Z_1, \phi Z_2) + c\eta(Z_1)\eta(Z_2), \tag{5.10}$$

where $a = -[\beta^2 - \alpha + \frac{4(\varepsilon+1)(\lambda+\beta\delta)}{4}]$, $b = 4\alpha\beta\delta$ and $c = \varepsilon\beta\delta - \mu$
and

$$QZ_1 = aZ_1 - b\phi Z_1 + c\eta(Z_1)\zeta, \tag{5.11}$$

which implies

$$r = 3a + \varepsilon c. \tag{5.12}$$

Proposition 3. *A $[(t\text{-}Sm)_3]$ on an η-"Ricci soliton" satisfying $R(U_1, U_2).M_p = 0$ is an Einstein like manifold with manifold of constant curvature.*

6. Parallel symmetric of (0, 2)-tensor field on [(t-Sm)]

Consider τ be a (0, 2)-tensor field which is presumed as parallel with regarding to Levi-Civita connection ∇, ie. $\nabla\tau = 0$. If Ricci identity is applied

$$\nabla^2\tau(U_1, U_2; U_3, W) - \nabla^2\tau(U_1, U_2; W, U_3) = 0,$$

we get [14]

$$\tau(R(U_1, U_2)U_3, W) + \tau(R(U_1, U_2)W, U_3) = 0. \tag{6.1}$$

Taking $U_3 = W = \zeta$ and using the symmetric property of τ, we have

$$\tau(R(U_1, U_2)\zeta, \zeta) = 0. \tag{6.2}$$

Suppose $(M, \phi, \zeta, \eta, g, \varepsilon)$ be an [(t-Sm)] with K-torse forming characteristic vector field. By (3.5) and (3.17) we obtain

$$2\varepsilon\alpha(a_1+\lambda)[\eta(X_1)\tau(\phi U_2,\zeta)-\eta(U_2)\tau(\phi U_1,\zeta)]+[(a_1+\lambda)^2-\alpha][\eta(U_1)\tau(U_2,\zeta)-\eta(U_2)\tau(U_1,\zeta)]=0. \tag{6.3}$$

Put $U_1 = \zeta$ in (6.3) we have

$$[(a_1+\lambda)^2-\alpha][\tau(U_2,\zeta)-\eta(U_2)\tau(\zeta,\zeta)]=0. \tag{6.4}$$

If $(a_1+\lambda)^2-\alpha \neq 0$, then

$$\tau(U_2,\zeta)=\eta(U_2)\tau(\zeta,\zeta). \tag{6.5}$$

Definition 6.2. *An [(t-Sm)] $(M, \phi, \zeta, \eta, g, \varepsilon)$ with K-torse-forming characteristic vector field is known as regular, provided $(a_1+\lambda)^2-\alpha \neq 0$.*

As τ is a parallel (0 , 2)-tensor field, therefore $\tau(\zeta,\zeta)$ is a constant. Applying the covariant derivative of (6.5) with regarding to U_1, we obtain

$$\tau(\nabla_{U_1}U_2,\zeta)-(a_1+\lambda)[\tau(U_1,U_2)-\eta(U_1)\eta(U_2)\tau(\zeta,\zeta)]-\varepsilon\alpha\tau(\phi U_1,U_2)=U_1(\eta(U_2))\tau(\zeta,\zeta), \tag{6.6}$$

by using $U_2 = \nabla_{U_1}U_2$ in (6.5), we obtain

$$\tau(\nabla_{U_1}U_2,\zeta)=\varepsilon g(\nabla_{U_1}U_2,\zeta)\tau(\zeta,\zeta), \tag{6.7}$$

substituting (6.7) in (6.6), we have

$$\tau(U_1,U_2)=\varepsilon g(U_1,U_2)\tau(\zeta,\zeta). \tag{6.8}$$

Theorem 6.4. *A symmetric parallel second order covariant tensor is regular [(t-Sm)] with K-torse forming characteristic vector field will be a constant multiple of the metric tensor.*

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