$g^{**}\Lambda$ - Closed Sets in Topological Spaces

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ABSTRACT: The aim of this paper is to introduce a new class of sets called $g^{**}\Lambda$-closed sets in topological spaces. Some basic properties of $g^{**}\Lambda$-closed sets are analyzed and obtained the interrelations between $g^{**}\Lambda$-closed sets and already existing closed sets.

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I.INTRODUCTION

Levine [5] introduced the notion of generalized closed sets in topological spaces. Veera Kumar [8] introduced g^*-closed sets in topological spaces. Maki [4] introduced the notion of Λ-sets in topological spaces. A Λ-set is a set A which is equal to its kernel i.e to the intersection of all open supersets of A. Arenas et al.[1] introduced λ-closed sets by using closed sets and Λ-sets. In this paper a new form of closed sets called $g^{**}\Lambda$-closed sets which contains the class of $g^*\Lambda$-closed sets and contained in the class of $g\Lambda$-closed sets in topological spaces is introduced.

II.PRELIMINARIES

Throughout this paper (X, τ) denotes non-empty topological space on which no separation axioms are assumed, unless otherwise mentioned. The closure and complement of a subset A of a space (X, τ) are denoted by cl(A) and A^c respectively.

Definition 2.1

A subset A of a topological space (X, τ) is called a

(1) Regular closed set [7] if A=cl(int(A))

(2) generalized closed set (g-closed) [5] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).

(3) g^*-closed set[8] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in (X, τ).

(4) λ- closed set [1] if $A = C \cap D$ where C is a Λ-set and D is a closed set.

(5) $g\Lambda$ -closed set [3] if $cl_{\lambda}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).

(6) Λ **- g** closed set [3] if $cl_{\lambda}(A) \subseteq U$ whenever $A \subseteq U$ and U is λ-open in (X, τ).

(7) $g^*\Lambda$ -closed set [6] if $cl_{\lambda}(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in (X, τ).

The complements of the above mentioned sets are called their respective open-sets.

(8) The intersection of all λ-closed sets containing A is called the λ - closure of A and is denoted by $cl_{\lambda}(A)$ [2]

(9) A subset A of a topological space (X, τ) is called a $*T_{1/2}$–space if every g-closed subset of (X, τ) is g*- closed in (X, τ). [8]

Remark 2.2

1. Every Λ -set is a λ - closed set [1]
2. Every open and closed sets are λ - closed sets [1]

III. $g^{**}\Lambda$ -CLOSED SETS

Definition 3.1 A subset A of a topological space (X, τ) is said to be a $g^{**}\Lambda$ -closed set if $cl_{\lambda}(A) \subseteq U$ whenever $A \subseteq U$ and U is g^*-open in (X, τ).

Proposition 3.2 Every λ -closed set in (X,τ) is $g^{**}\Lambda$ -closed but not conversely.

Proof: Let A be a λ-closed set of (X, τ). Let U be any g^*-open set containing A in X. Since A is λ-closed, $cl_{\lambda}(A) = A \subseteq U$. Therefore A is $g^{**}\Lambda$ - closed.

Example 3.3 Let $X = \{a, b, c\}$ with $\tau = \{X, \varphi, \{a\}\}$. Then the subset {a, b} is $g^{**}\Lambda$ -closed but not λ-closed in (X, τ).

Proposition 3.4 Every closed set in (X, τ) is $g^{**}\Lambda$ -closed but not conversely.

Proof: The proof follows from remark 2.2 and proposition 3.2.

Example 3.5 Let $X = \{a, b, c\}$ with $\tau = \{X, \varphi, \{a, b\}\}$. Then the subset {a, c} is $g^{**}\Lambda$ -closed but not closed in (X, τ).

Proposition 3.6 Every open set in (X, τ) is $g^{**}\Lambda$ -closed but not conversely.

Proof: The proof follows from remark 2.2 and proposition 3.2.

Example 3.7 Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b, c\}\}$. Then the subset {a, b} is $g^{**}\Lambda$ -closed but not open in (X, τ).

Proposition 3.8 Every regular closed set in (X, τ) is $g^{**}\Lambda$ -closed but not conversely.

Proof: The proof follows from the fact that every regular closed set is closed and by proposition 3.4.

Example 3.9 Let $X=\{a,b,c\}$ with $\tau=\{X,\phi,\{a\},\{b\},\{a,b\}\}$. Then the subset {a} is $g^{**}\Lambda$ -closed but not regular closed in (X, τ).

Proposition 3.10 Every Λ-set in (X, τ) is $g^{**}\Lambda$-closed but not conversely.

Proof: The proof follows from remark 2.2 and proposition 3.2.

Example 3.11 Let $X=\{a,b,c\}$ with $\tau=\{X,\varphi,\{a\}\}$. Then the subset {b, c} is $g^{**}\Lambda$ -closed but not a Λ-set in (X, τ).

Proposition 3.12 Every g^*-closed set in (X, τ) is $g^{**}\Lambda$ -closed but not conversely.

Proof: Let A be a g^*- closed set of (X, τ). Let U be any g^*-open set containing A in X. Since every g^*-open set is g-open and A is g^*-closed set, $cl(A)\subseteq U$. Since every closed set is λ- closed, $cl_\lambda(A)\subseteq cl(A))\subseteq$ U. Hence A is $g^{**}\Lambda$-closed.

Example 3.13 Let $X=\{a,b,c\}$ with $\tau=\{X,\varphi,\{a,b\}\}$. Then the subset {a, b} is $g^{**}\Lambda$-closed but not g^*-closed in (X, τ).

Proposition 3.14 Every $g^*\Lambda$-closed set in (X, τ) is $g^{**}\Lambda$ -closed but not conversely.

Proof: Let A be a $g^*\Lambda$- closed set of (X, τ). Let U be any g^*-open set containing A in X. Since every g^*-open set is g-open and A is $g^*\Lambda$-closed set, $cl_\lambda(A)\subseteq U$. Hence A is $g^{**}\Lambda$-closed.

Example 3.15 Let $X=\{a,b,c\}$ with $\tau=\{X,\phi,\{a\}\}$. Then the subset {a, b} is $g^{**}\Lambda$-closed but not $g^*\Lambda$-closed in (X, τ).

Proposition 3.16 Every $g^{**}\Lambda$-closed set in (X,τ) is $g\Lambda$-closed but not conversely.

Proof: Let A be a $g^{**}\Lambda$- closed set of (X ,τ). Let U be any open set containing A in X. Since every open set is g^*-open and A is $g^{**}\Lambda$-closed set, $cl_\lambda(A)\subseteq U$. Hence A is $g\Lambda$-closed.

Example 3.17 Let $X=\{a,b,c\}$ with $\tau=\{X,\varphi,\{a,b\}\}$. Then the subset {a} is $g\Lambda$ -closed but not $g^{**}\Lambda$-closed in (X, τ).

Definition 3.18 A subset A of a topological space (X, τ) is said to be $g^{**}\Lambda$-open if its complement A^c is $g^{**}\Lambda$-closed.

Proposition 3.19 Every λ-open (respectively open, regular open, Λ, g^*-open and $g^*\Lambda$ -open) set is $g^{**}\Lambda$-open. Every $g^{**}\Lambda$-open set is $g\Lambda$-open

Proposition 3.20 Every closed set in (X, τ) is $g^{**}\Lambda$ -open.

Proof: Let A be a closed set of (X, τ). Then A^c is open. By remark 2.2, A^c is λ-closed which implies A is λ-open. By proposition 3.19, A is $g^{**}\Lambda$ -open.

Remark 3.21 From the relations we observed that $g^{**}\Lambda$-closed sets and $g^{**}\Lambda$ -open sets are stronger forms of $g\Lambda$ -closed sets and $g\Lambda$ -open sets respectively and weaker forms of $g^*\Lambda$- closed sets and $g^*\Lambda$- open sets respectively.

Remark 3.22 The following example shows that the intersection of any two $g^{**}\Lambda$-closed sets need not be a $g^{**}\Lambda$-closed set.

Example 3.23 Let $X = \{a,b,c\}$ with $\tau = \{X,\varphi,\{a,b\}\}$. Then the subsets {a, b} and {b, c} are $g^{**}\Lambda$-closed sets but their intersection {b} is not $g^{**}\Lambda$-closed in (X, τ).

Remark: 3.24 $g^{**}\Lambda$-closed sets will not form a topology, as intersection of $g^{**}\Lambda$-closed sets is not a $g^{**}\Lambda$-closed set.

Remark 3.25 $g^{**}\Lambda$-closed sets and Λ**-g**-closed sets are independent as seen from the following examples.

Example 3.26 Let $X = \{a,b,c\}$ with $\tau = \{X,\varphi,\{a,b\}\}$. Then the subset {a} is Λ**-g** -closed but not $g^{**}\Lambda$-closed in (X, τ).

Example 3.27 Let $X = \{a,b,c\}$ with $\tau = \{X,\varphi,\{a\},\{a,b\}\}$. Then the subset {a, c} is $g^{**}\Lambda$-closed but not Λ**-g** -closed in (X, τ).

Remark 3.28 $g^{**}\Lambda$-closed sets and g-closed sets are independent.

Proposition 3.29 In $^*T_{1/2}$–space every g- closed set is $g^{**}\Lambda$-closed set.

Proof: Let A be a g- closed set of (X, τ). Since (X, τ) is a $^*T_{1/2}$ - space, A is g^*-closed. By proposition 3.12 we have A is $g^{**}\Lambda$-closed.

IV. Properties of $g^{**}\Lambda$ -closed sets

Theroem 4.1 Let A be a $g^{**}\Lambda$-closed subset of (X, τ). Then $cl_\lambda(A) - A$ contains no non-empty closed set in X.

Proof: Suppose that A is $g^{**}\Lambda$-closed. Let F be a non empty closed subset of $cl_\lambda(A) - A$. Then F^c is open and hence g^*-open such that $A \subseteq F^c$. Since A is a $g^{**}\Lambda$-closed set, $cl_\lambda(A) \subseteq F^c$. Thus

$F \subseteq (cl_{\lambda}(A))^c$. Since every closed set is λ-closed, F is λ-closed. Hence $F \subseteq cl_{\lambda}(A)$. Therefore $F \subseteq [(cl_{\lambda}(A))^c \cap cl_{\lambda}(A)] = \varphi$. Hence $F=\phi$.

Remark 4.2 The converse of the above theorem is not true as seen from the following example.

Example 4.3 Let $X = \{a, b, c\}$ with $\tau = \{X, \varphi, \{a, b\}\}$. If A={b} then $cl_{\lambda}(A) - A$ ={a, b}-{b}={a} does not contain non-empty closed set. However A is not $g^{**}\Lambda$-closed.

Theroem 4.4 If a subset A is $g^{**}\Lambda$-closed in (X, τ), then $cl_{\lambda}(A) - A$ contains no non-empty g^*-closed set.

Proof: Let A be a $g^{**}\Lambda$-closed set. Let F be a g^*-closed set contained in $cl_{\lambda}(A) - A$. Then F^c is a g^*-open set in X such that $A \subseteq F^c$. Since A is a $g^{**}\Lambda$-closed set of X, $cl_{\lambda}(A) \subseteq F^c$. Thus $F \subseteq (cl_{\lambda}(A))^c$. Also $F \subseteq cl_{\lambda}(A) - A$. Therefore $F \subseteq [(cl_{\lambda}(A))^c \cap cl_{\lambda}(A)] = \varphi$. Hence $F = \varphi$.

Proposition 4.5 If a subset A is g^*-open and $g^{**}\Lambda$-closed set in (X, τ). Then A is a λ-closed set of X.

Proof: Since A is g^*-open and $g^{**}\Lambda$-closed, $cl_{\lambda}(A) \subseteq A$. Hence A is λ-closed.

Theorem 4.6 Let A be $g^{**}\Lambda$-closed and g^*-open in (X,τ). If G is λ-closed in (X,τ), then $A \cap G$ is $g^{**}\Lambda$-closed.

Proof: Since A is $g^{**}\Lambda$-closed and g^*-open, A is λ-closed by proposition 4.5. Therefore if G is λ-closed in X, then $A \cap G$ is λ-closed in X, as the intersection of λ-closed sets is a λ-closed set. Hence by proposition 3.2, $A \cap G$ is $g^{**}\Lambda$-closed

Theorem 4.7 If A is a $g^{**}\Lambda$-closed set in (X,τ) and $A \subseteq B \subseteq cl_{\lambda}(A)$. Then B is also a $g^{**}\Lambda$-closed set.

Proof: Let U be a g^*-open set of (X,τ) such that $B \subseteq U$. Then $A \subseteq U$. Since A is a $g^{**}\Lambda$-closed set, $cl_{\lambda}(A) \subseteq U$. Also since $B \subseteq cl_{\lambda}(A)$, $cl_{\lambda}(B) \subseteq cl_{\lambda}(cl_{\lambda}(A)) = cl_{\lambda}(A)$. Hence $cl_{\lambda}(B) \subseteq U$. Therefore B is also a $g^{**}\Lambda$-closed set in (X, τ).

Theorem 4.8 Let A be a $g^{**}\Lambda$-closed set in (X,τ). Then A is λ-closed if and only if $cl_{\lambda}(A) - A$ is g^*-closed.

Proof:(Necessity) Let A be a λ-closed subset of (X,τ).Then $cl_\lambda(A) = A$ and therefore $cl_\lambda(A) - A = \varphi$ which is g^*-closed.

Sufficiency: Let $cl_\lambda(A) - A$ be a g^*-closed set. Since A is $g^{**}\Lambda$-closed by theorem 4.4 $cl_\lambda(A) - A$ contains no non- empty g^*-closed set which implies $cl_\lambda(A) - A = \varphi$ That is $cl_\lambda(A) = A$. Therefore A is λ-closed.

Theorem 4.9 For each $x \in (X,\tau)$ either $\{x\}$ is g^*-closed or X-{x} is $g^{**}\Lambda$-closed in (X,τ).

Proof: Let $x \in X$ and suppose that {x} is not g^*- closed in X. Then X-{x} is not g^*-open in X. Hence X is the only g^*-open set containing X-{x}. That is $(X\text{-}\{x\}) \subseteq X$. Hence $cl_\lambda(X\text{-}\{x\}) \subseteq X$ which implies that X-{x} is $g^{**}\Lambda$-closed in (X,τ).

Definition 4.10 The intersection of all g^*-open subsets of (X,τ) containing A is called g^*-kernal of A and is denoted by g^*-ker(A)

i.e $g^*\text{-ker}(A) = \cap\{U / U$ is g^*-open in (X,τ) and $A \subseteq U\}$

Theorem 4.11 A subset A of (X,τ) is $g^{**}\Lambda$-closed if and only if $cl_\lambda(A) \subseteq g^*\text{-ker}(A)$.

Proof: suppose that A is $g^{**}\Lambda$-closed in (X,τ).Then $cl_\lambda(A) \subseteq U$ whenever $A \subseteq U$ and U is g^*- open in (X,τ). Let $x \in cl_\lambda(A)$. If $x \notin g^*\text{-ker}(A)$, then there exists a set $U \in g^*O(X,\tau)$ such that $x \notin U$ and $A \subseteq U$. Since U is a g^*- open set containing A, we have x not belongs to $cl_\lambda(A)$, a contradiction.

Conversely, let $cl_\lambda(A) \subseteq g^*\text{-ker}(A)$. If U is any g^*-open set containing A, then $cl_\lambda(A) \subseteq g^*\text{-ker}(A)$ we have $cl_\lambda(A) \subseteq U$. Hence A is $g^{**}\Lambda$-closed.

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